Fluctuation and interaction induced instability of dark solitons in Bose-Einstein condensates

Arko Roy

Physical Research Laboratory, Ahmedabad

July 3, 2014

Plan of the talk

Introduction

$T \neq 0$ regime

Generalized GP equation

Solitons

Binary BEC

Conclusion

(Ideal) Bose-Einstein Condensation

- Macroscopic occupation of non-interacting bosons in the ground state of the system
- A gas of bosonic particles cooled below a critical temperature T_c condenses into an ideal Bose-Einstein condensate (BEC)
- Criteria for condensation

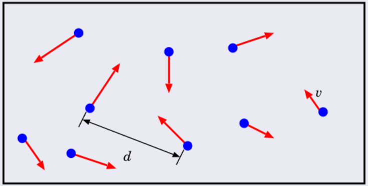
$$\varpi = n\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2} = 2.612,$$

- De Broglie wavelength λ_{dB} comparable to the distance between the particles–wave packets start to overlap

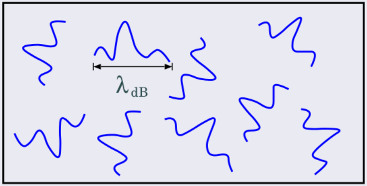
Anderson et al., *Science* **269**, (1995); Davies et al., *Phys. Rev. Lett* **75**, (1995); Ketterle et al., *Rev. Mod. Phys.* **74**, (2002).

Basic Phenomenon

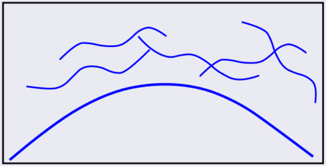
High Temperature (T), Density d^{-3}



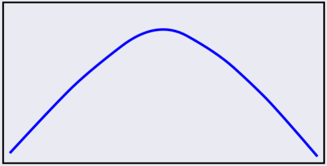
Low Temperature (T), λ_{dB}



Bose-Einstein Condensate, $T = T_{crit}$



Pure Bose Condensate, $T = 0$



P. Muruganandam (Workshop on HPC, PRL Ahmedabad, 2012).

General Criteria for BEC

When interactions are present$\Rightarrow$ Single-particle energy levels are not defined. A *reduced single-particle density operator* is defined

$$\hat{\rho}_1 \equiv \mathrm{Tr}_{2,3,\cdots N}\, \hat{\rho}$$

where $\mathrm{Tr}_{2,3,\cdots N}$ $\rightarrow$Trace of $\hat{\rho}$ w.r.t particles $2, 3, \cdots N$

- Define $\hat{\sigma}_1 = N\hat{\rho}_1$
- **Penrose-Onsager condition**:

$$\frac{n_M}{N} = e^{\mathcal{O}(1)}$$

- n_M: largest eigenvalue of $\hat{\sigma}_1$, condensation occurs in corresponding eigenstate
- $e^{\mathcal{O}(1)}$: positive number of the order of unity.

O. Penrose and L. Onsager, *Phys. Rev*, **104**, (1956)

Gross-Pitaevskii equation

- Equation of motion of the condensate wavefunction is given by Gross-Pitaevskii equation(GPE), strictly valid at $T = 0$K.

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm trap}(\mathbf{r}) + gN|\psi|^2\right]\psi,$$

  - $\psi \equiv \psi(\mathbf{r}, t)$: condensate wave function
  - $g = \dfrac{4\pi\hbar^2 a}{m}$
  - a: atomic scattering length > 0 : repulsive
  - N: Number of atoms in the condensate

$$V_{\rm trap} = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

E. P. Gross, *Il Nuovo Cimento Series 10*, **20**, (1961);
L. P. Pitaevskii, *Soviet Physics JETP-USSR*, **13**, (1961);
C. Pethick & H. Smith, *Bose-Einstein Condensation in Dilute Gases*, (2008)

Why do we study finite temperature effects?

Region of interest :: $0 < T < T_c$

- $T = 0K$ is physically unattainable. Experiments take place at finite temperatures.
- When $T \neq 0$, the condensate co-exists with the *thermal cloud*. Interactions between condensate and non-condensate(thermal) atoms cannot be neglected.

Why do we study finite temperature effects?

Region of interest :: $0 < T < T_c$

- $T = 0K$ is physically unattainable. Experiments take place at finite temperatures.
- When $T \neq 0$, the condensate co-exists with the *thermal cloud*. Interactions between condensate and non-condensate(thermal) atoms cannot be neglected.

Modify $T = 0$ GPE to include effects of temperature.

Many-body Hamiltonian

$$\hat{H} = \int \underbrace{d\mathbf{r}\, \hat{\psi}^{\dagger}(\mathbf{r},t) \left[\hat{h}(\mathbf{r}) - \mu\right] \hat{\psi}(\mathbf{r},t)}_{\text{single-particle part}}$$

$$+\frac{1}{2} \iint \underbrace{d\mathbf{r} d\mathbf{r}'\, \hat{\psi}^{\dagger}(\mathbf{r},t)\hat{\psi}^{\dagger}(\mathbf{r}',t) U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t)}_{\text{two-particle interaction term}}$$

where $\hat{h} = K.E + V_{\mathrm{trap}}$

$$U(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}'), \left\langle \int d\mathbf{r}\, \hat{\psi}^{\dagger}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) \right\rangle = N$$

U:: Repulsive contact interaction; N:: Total number of atoms

$$\left[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\right] = \left[\hat{\psi}^{\dagger}(\mathbf{r}), \hat{\psi}^{\dagger}(\mathbf{r}')\right] = 0; \left[\hat{\psi}(\mathbf{r}), \hat{\psi}^{\dagger}(\mathbf{r}')\right] = \delta(\mathbf{r}-\mathbf{r}')$$

A.Griffin, *Phys. Rev. B*, **53**, (1996)

Hartree-Fock-Bogoliubov(HFB) approximation

The Bose field operator is

$$\hat{\psi}(\mathbf{r},t) = \sum_{i=0} \hat{\alpha}_i(t)\psi_i(\mathbf{r}) = \hat{\alpha}_0(t)\psi_0(\mathbf{r}) + \sum_{i=1} \hat{\alpha}_i(t)\psi_i(\mathbf{r}),$$

$$\hat{\alpha}_i^\dagger|n_0 n_1,\cdots,n_i,\cdots\rangle = \sqrt{(n_i+1)}|n_0 n_1,\cdots,n_i+1,\cdots\rangle,$$

$$\hat{\alpha}_i|n_0 n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i}|n_0 n_1,\cdots,n_i-1,\cdots\rangle.$$

BEC occurs when: $n_0 \equiv N_0 \gg 1 \rightarrow N_0, N_0 \pm 1 \approx N_0$
where $N_0 \rightarrow$ Number of **condensate** atoms

HFB approximation: $\hat{\alpha}_0 = \hat{\alpha}_0^\dagger = \sqrt{N_0}$, then

$$\hat{\psi}(\mathbf{r},t) = \sqrt{N_0}\psi_0(\mathbf{r})e^{-i\mu t/\hbar} + \tilde{\psi}(\mathbf{r},t),$$

such that, $\langle\tilde{\psi}(\mathbf{r},t)\rangle = \langle\tilde{\psi}^\dagger(\mathbf{r},t)\rangle = 0$.

Dalfovo et al, *Rev. Mod. Phys.*,**71**, (1999)

Generalized GP equation

Equation of motion of the Bose field operator

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{r},t)}{\partial t} = (\hat{h}-\mu)\hat{\psi}(\mathbf{r},t) + g\hat{\psi}^{\dagger}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)$$

where, $\hat{\psi}(\mathbf{r},t) = \phi(\mathbf{r}) + \tilde{\psi}(\mathbf{r},t)$, and $\phi(\mathbf{r}) = \sqrt{N_0}\psi_0(\mathbf{r})$. $\phi/\tilde{\psi}$ is the condensate/non-condensate part.

$$\tilde{\psi}^{\dagger}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) \simeq 2\underbrace{\langle\tilde{\psi}^{\dagger}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle}_{\tilde{n}}\tilde{\psi}(\mathbf{r},t) + \underbrace{\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle}_{\tilde{m}}\tilde{\psi}^{\dagger}(\mathbf{r},t)$$

$\tilde{n} \rightarrow$ Non-condensate density; $\tilde{m} \rightarrow$ Anomalous average

Generalized GPE

Including the thermal component and anomalous term, the generalized GP equation is

$$(\hat{h} - \mu)\phi(\mathbf{r}) + g|\phi(\mathbf{r})|^2\phi(\mathbf{r}) + \underbrace{2g\tilde{n}(\mathbf{r})\phi(\mathbf{r}) + g\tilde{m}(\mathbf{r})\phi^*(\mathbf{r})}_{T-dependent} = 0$$

- $\hat{h} = K.E. + V_{\mathrm{trap}}$
- $\int |\phi(\mathbf{r})|^2 \, d\mathbf{r} = 1$

A.Griffin, *Phys. Rev. B*, **53**, (1996);
D. A. W. Hutchinson, E. Zaremba, and A. Griffin *Phys. Rev. Lett.*, **78**, (1997)

Bogoliubov de-Gennes equations

Equation of motion of the thermal component

$$
\begin{aligned}
i\hbar\frac{\partial\tilde{\psi}}{\partial t} &= i\hbar\frac{\partial}{\partial t}(\hat{\psi}-\phi), \\
&= (\hat{h}-\mu)\tilde{\psi} + 2gn(\mathbf{r})\tilde{\psi} + gm(\mathbf{r})\tilde{\psi}^{\dagger},
\end{aligned}
$$

where, $n(\mathbf{r}) = |\phi(\mathbf{r})|^2 + \tilde{n}(\mathbf{r})$; $m(\mathbf{r}) = \phi^2(\mathbf{r}) + \tilde{m}(\mathbf{r})$;
$\tilde{\psi} = \sum_j \left[u_j\hat{\alpha}_j e^{-iE_j t} - v_j^*\hat{\alpha}_j^{\dagger} e^{iE_j t}\right]$;
$u_j, v_j \Rightarrow$ quasiparticle amplitudes
Bogoliubov de-Gennes equations:

$$
\begin{aligned}
\mathcal{L}u_j &- gmv_j = E_j u_j \\
\mathcal{L}v_j &- gm^* u_j = -E_j v_j
\end{aligned}
$$

where $\mathcal{L} = \hat{h} - \mu + 2gn(\mathbf{r})$

Non-condensate density

Density of the thermal component:

$$\langle \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle = \tilde{n} = \sum_j \left\{ \left[|u_j|^2 + |v_j|^2\right] \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j\rangle + |v_j|^2 \right\}.$$

and multiplying factor

$$\langle \hat{\alpha}_j^\dagger \hat{\alpha}_j\rangle = \frac{1}{e^{\beta E_j} - 1} \equiv N_0(E_j).$$

is the Bose-Einstein distribution.

At $T = 0$**,** $\tilde{n} = \sum_j |v_j|^2 \rightarrow$ **Quantum depletion**

The anomalous average:

$$\langle \tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle = \tilde{m} = -\sum_j u_j v_j^* \left[2\langle \hat{\alpha}_j^\dagger \hat{\alpha}_j\rangle + 1\right],$$

A.Griffin, *Phys. Rev. B*, **53**, (1996)

Summary of steps

I. **Generalized GPE**:

$$(\hat{h}-\mu)\phi(\mathbf{r})+g|\phi(\mathbf{r})|^2\phi(\mathbf{r})+2g\tilde{n}(\mathbf{r})\phi(\mathbf{r})+g\tilde{m}\phi^*(\mathbf{r})=0$$

II. **Bogoliubov de-Gennes equations**:

$$\begin{aligned}\mathcal{L}u_j &- gmv_j = E_j u_j\\ \mathcal{L}v_j &- gm^* u_j = -E_j v_j\end{aligned}$$

where $\mathcal{L}=\hat{h}-\mu+2g(|\phi(\mathbf{r})|^2+\tilde{n})$

III. **Non-condensate density**:

$$\begin{aligned}\tilde{n}(\mathbf{r}) &= \sum_j \left\{\left[|u_j|^2+|v_j|^2\right]\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle+|v_j|^2\right\}\\ \tilde{m}(\mathbf{r}) &= -\sum_j u_j v_j^*[2\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle+1]\end{aligned}$$

Problems in HFB

- HFB theory is not *gapless*. Violates Hugenholtz-Pines theorem.
  Reason :: Approximate factorization of operator averages
- The anomalous pair average $\tilde{m}$ is divergent.
  Reason :: Inconsistent treatment of collisions through contact potential. Treats collisions of different energy with same probability.

Gapless finite temperature approximation

$\Downarrow$

Neglect $\tilde{m}$.

$\Downarrow$

HFB-Popov approximation

Valid in $0 < T \lesssim 0.65\, T_c$

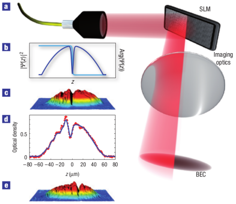
N. P. Proukakis and B. Jackson, *J. Phys. B*, **41**, (2008)

Solitons in BEC

- Localized disturbances which propagate without change of form. Subject of intensive study in nonlinear optics describing the propagation of light pulses in optical fibers.
  Ultracold atoms:
- At $T = 0$, 1D GPE predicts a stable dark (bright) solitonic solution when inter-atomic interactions are repulsive (attractive).
- Features of dark soliton
  - Local density minimum and is equal to zero,
  - Sharp phase gradient of π across the position of minimum of the wave function.
  - Presence of an *anomalous mode* – signature of an energetically excited state.
- At $T \neq 0$, dark soliton exhibits *dynamical* instabilities.

C. Pethick & H. Smith, *Bose-Einstein Condensation in Dilute Gases*, (2008)

Solitons in BEC

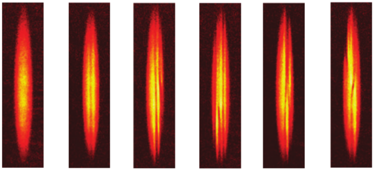


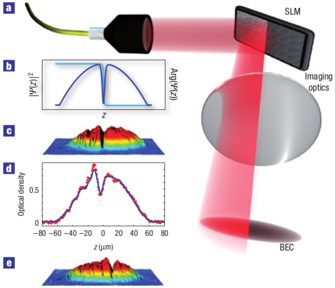
- Features of dark soliton
  - Local density minimum and is equal to zero,
  - Sharp phase gradient of π across the position of minimum of the wave function.
  - Presence of an *anomalous mode* – signature of an energetically excited state.
- At $T \neq 0$, dark soliton exhibits *dynamical* instabilities.

C. Pethick & H. Smith, *Bose-Einstein Condensation in Dilute Gases*, (2008)

Formation of soliton

- Spontaneously created during the formation of quasi-1D BECs through Kibble-Zurek mechanism for $\approx 10^6$ Sodium (Na) atoms. `Ref:` Lamporesi et. al, Nat. Phys. **9** (2013).
- *Phase-imprinting* employed to create solitons in elongated BECs for $\approx 10^5$ Rubidium (^{87}Rb) atoms.
  `Ref:` Burger et. al, PRL **83** (1999); Becker et. al, Nat. Phys. **4** (2008).

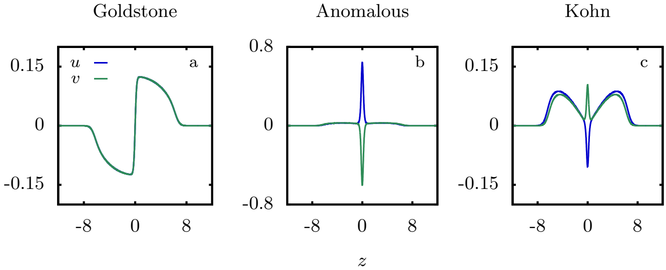




Anomalous mode

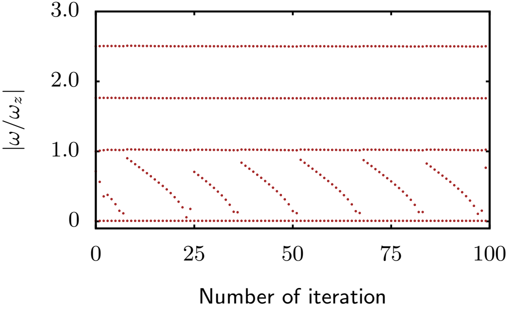
Krein sign $\Delta_j = \int dz \, (|u_j|^2 - |v_j|^2) E_j$

- Negative *Krein sign* implies presence of *anomalous mode*. Signature of energetic instability.
- When modes with opposite *Krein sign* collide, it gives rise to complex eigenfrequencies. Signature of dynamical instability.

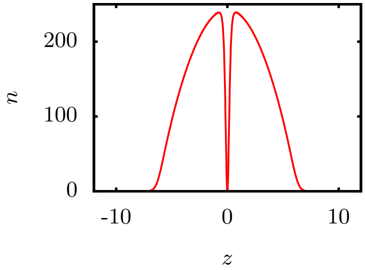


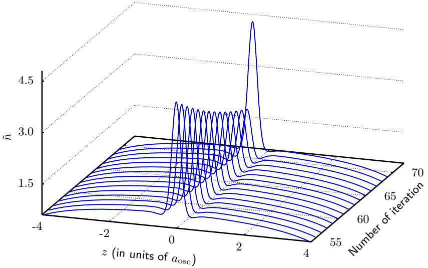
Middelkamp et. al, *Phys. Rev. A*, **81**, (2010)

Fluctuation induced instability: $T = 0$ results

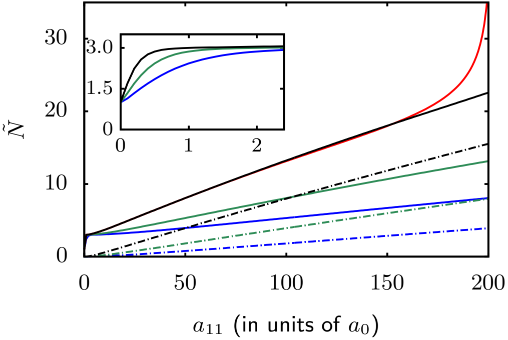


$a_{^{87}\mathrm{Rb}} = 100 a_0$
$N_{^{87}\mathrm{Rb}} = 2\,000$
$\omega_{z\,^{87}\mathrm{Rb}} = 2\pi \times 4.55\,\mathrm{Hz}$
$\omega_\perp = 20\omega_z$





Quantum Depletion at $T = 0$



$\tilde{N} = \int_{-\infty}^{\infty} \tilde{n}\, dz$, N=500(Blue), 1000(Green), 2000(Black)

Solid lines→ Presence of soliton,

Dashed lines→ Absence of soliton,

AR, D. Angom , arXiv : 1405:6459

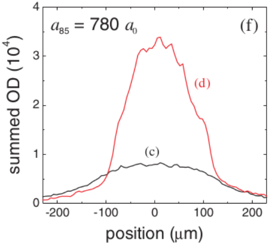
Unique feature of binary BEC

Role of interactions

Phase Separation

$U_{11} U_{22} - (U_{12})^2 > 0$

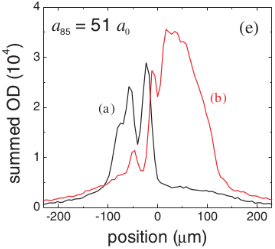
Miscible regime



^{85}Rb–^{87}Rb

$U_{11} U_{22} - (U_{12})^2 < 0$

Immiscible regime



Papp et. al, *Phys. Rev. Lett.*,**101**, (2008)

Experimental realization of binary BEC

2 different **atoms**

- ^{87}Rb–^{41}K
  Thalhammer et. al, PRL, **100**, (2008)
- ^{84}Sr–^{87}Rb
  Pasquiou et. al, PRA, **88**, (2013)
- ^{23}Na–^{87}Rb
  Xiong et. al, arXiv:1305.7091, (2013)
- ^{133}Cs–^{87}Rb
  McCarron et. al, PRA(R), **84**, (2011)

2 different **isotopes**

- ^{85}Rb–^{87}Rb
  Papp et. al, PRL, **101**, (2008)

2 different **hyperfine states**

- $|F=1, m_F=+1\rangle$, $|F=2, m_F=-1\rangle$ of ^{87}Rb
  $|F=2, m=2\rangle, |F=2, m=-1\rangle$
  Tojo et. al, PRA, **82**, (2010)

Coupled Generalized GP equation

From **HFB-Popov** approximation

$$\begin{aligned}
\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} &= \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix},\\
\hat{h}_1\phi_1 &+ U_{11}\left[n_{1c}+2\tilde{n}_1\right]\phi_1 + U_{12}[n_{2c}+\tilde{n}_2]\phi_1 = 0,\\
\hat{h}_2\phi_2 &+ U_{22}\left[n_{2c}+2\tilde{n}_2\right]\phi_2 + U_{12}[n_{1c}+\tilde{n}_1]\phi_2 = 0.
\end{aligned}$$

- $n_{kc} = |\phi_k|^2$: Condensate density of k^{th} species
- $\tilde{n}_k$: Non-condensate density of k^{th} species

where, $\hat{h}_k = -\frac{\hbar^2\nabla^2}{2m_k} + V_k(\mathbf{r}) - \mu_k$

BdG Equations

$$
\begin{aligned}
\hat{\mathcal{L}}_1 u_{1j} - U_{11}\phi_1^2 v_{1j} + U_{12}\phi_1(\phi_2^* u_{2j} - \phi_2 v_{2j}) &= \hbar E_j u_{1j}, \\
-\hat{\mathcal{L}}_1 v_{1j} + U_{11}\phi_1^{*2} u_{1j} - U_{12}\phi_1^*(\phi_2 v_{2j} - \phi_2^* u_{2j}) &= \hbar E_j v_{1j}, \\
\hat{\mathcal{L}}_2 u_{2j} - U_{22}\phi_2^2 v_{2j} + U_{12}\phi_2(\phi_1^* u_{1j} - \phi_1 v_{1j}) &= \hbar E_j u_{2j}, \\
-\hat{\mathcal{L}}_2 v_{2j} + U_{22}\phi_2^{*2} u_{2j} - U_{12}\phi_2^*(\phi_1 v_{1j} - \phi_1^* u_{1j}) &= \hbar E_j v_{2j},
\end{aligned}
$$

$$
\begin{aligned}
\text{where, } \hat{\mathcal{L}}_1 &= \left(\hat{h}_1 + 2U_{11}n_1 + U_{12}n_2\right), \\
\hat{\mathcal{L}}_2 &= \left(\hat{h}_2 + 2U_{22}n_2 + U_{12}n_1\right), \\
n_1 &= n_{1c} + \tilde{n}_1, n_2 = n_{2c} + \tilde{n}_2.
\end{aligned}
$$

$$
\tilde{\psi}_k(\mathbf{r}, t) = \sum_j \left[u_{kj}(\mathbf{r})\hat{\alpha}_j(\mathbf{r})e^{-iE_j t} - v_{kj}^*(\mathbf{r})\hat{\alpha}_j^\dagger(\mathbf{r})e^{iE_j t}\right]
$$

Non-Condensate density:

$$
\tilde{n}_k = \sum_j \left\{\left[|u_{kj}|^2 + |v_{kj}|^2\right] N_0(E_j) + |v_{kj}|^2\right\}
$$

AR, S. Gautam, D. Angom, *Phys. Rev. A.*,**89**, (2014)

Soliton in binary BEC

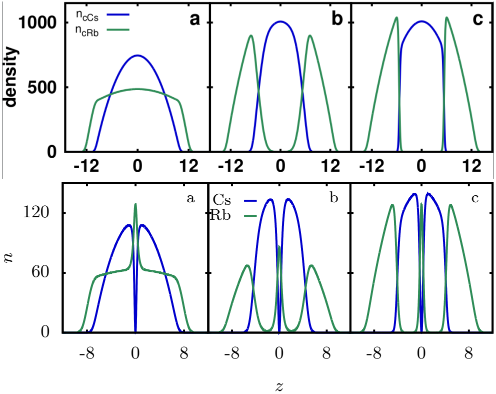
Density profiles

from miscible to immiscible(phase-separated)

$^{133}\mathrm{Cs} - ^{87}\mathrm{Rb}$ mixture in quasi-1D trap

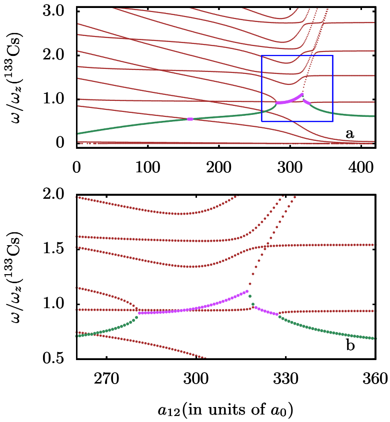
$(\omega_{z(\mathrm{Rb})} = 2\pi \times 3.89\mathrm{Hz}, \omega_{z(\mathrm{Cs})} = 2\pi \times 4.55\mathrm{Hz}, \omega_{\perp} = 30\omega_z)$

$a_{\mathrm{CsCs}} = 280a_0, a_{\mathrm{RbRb}} = 100a_0$

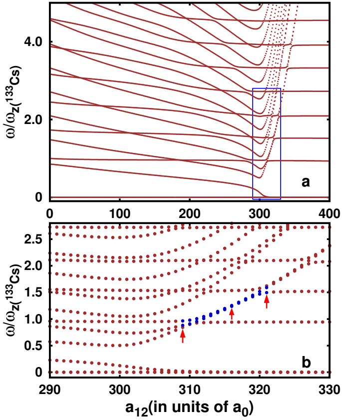


Interaction induced instability : $T = 0$ results

Mode evolution



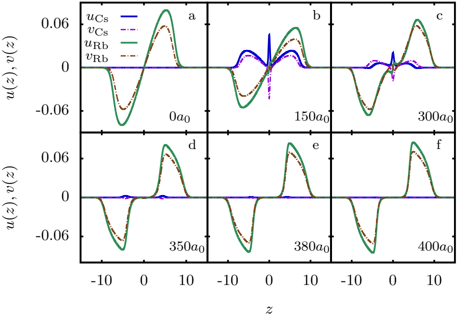
TBEC with soliton

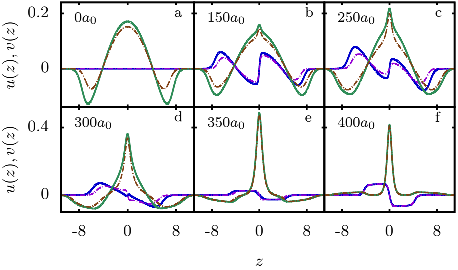


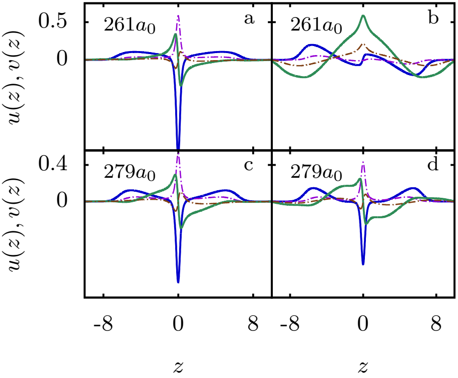
TBEC without soliton

AR, S. Gautam, D. Angom, *Phys. Rev. A.*,**89**, (2014)
AR, D. Angom , arXiv : 1405:6459

Metamorphosis: From Bogolon to Goldstone







Conclusion

- We have predicted *fluctuation induced instability* due to dark soliton in BECs at $T = 0$.
- We have also shown presence of soliton enhances the quantum depletion.
- We have generalized HFB-Popov approximation to analyze finite temperature effects on binary mixtures of Bose condensed gases.
- Symmetry preserving solution of highly phase separated condensates with soliton gives rise to a fourth additional Goldstone mode. Earlier, the presence of a third Goldstone mode was also predicted in highly immiscible binary BEC without soliton.
- Binary BEC with soliton in one of the components give rise to *interaction induced instability*.

Conclusion

- We have predicted *fluctuation induced instability* due to dark soliton in BECs at $T = 0$.
- We have also shown presence of soliton enhances the quantum depletion.
- We have generalized HFB-Popov approximation to analyze finite temperature effects on binary mixtures of Bose condensed gases.
- Symmetry preserving solution of highly phase separated condensates with soliton gives rise to a fourth additional Goldstone mode. Earlier, the presence of a third Goldstone mode was also predicted in highly immiscible binary BEC without soliton.
- Binary BEC with soliton in one of the components give rise to *interaction induced instability*.

THANK YOU